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ACCOUNTING FOR THE ELASTIC PROPERTIES OF A NON-NEWTONIAN MATERIAL UNDER ITS VISCOSIMETRIC FLOW

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This paper considers the deformation and viscoplastic flow of a non-Newtonian material enclosed between coaxial rigid cylindrical surfaces, each of which performs a rotation followed by a stop and a rotation in the opposite direction. The problem is solved using the model of large elastoviscoplastic deformations, in contrast to the classical solutions obtained using the model of a rigid viscoplastic body. The parameters of the viscosimetric process are calculated in both the region of viscoplastic flow developed and the region of elastic deformation.

Key words: *elasticity, viscoplasticity, large strains, residual stresses.*

In models of viscous and viscoplastic media, constants are usually determined by the properties of viscosimetric flows using the exact solution of the corresponding boundary-value problem. In the theory of viscous and viscoplastic media, such solutions are classical [1–3]. A problem of current issue is the investigation of effects due to the elastic properties of intensely deformed materials. Among such effects are inadmissible changes in the geometry of articles subjected to pressure treatment during manufacture (elastic afteraction), leading to the occurrence of inadmissible large residual stresses. Calculations with similar effects taken into account should be performed using the mathematical model of large elastoviscoplastic deformations. In this case, it is necessary to have exact solutions of the equations of this model not only for processing experimental results but also for testing the calculation algorithms. In the present paper, such a solution is constructed.

1. Basic Model Relations. The problem is solved using the model of large elastoplastic deformations proposed in [4] and extended to the case of deformation with allowance for viscosity [5]. In the Cartesian rectangular system of spatial Eulerian coordinates x_i, the reversible (elastic) e_{ij} and irreversible (plastic) p_ij components (undeterminable in experiments) of the total Almansi strain tensor p_{ij} are defined by the differential equations of variation (transfer) in the form

$$\frac{De_{ij}}{Dt} = \varepsilon_{ij} - \varepsilon^p_{ij} - \frac{1}{2}\,(e_{ik}(\varepsilon_{kj} - \varepsilon^p_{kj} - z_{kj}) + (\varepsilon_{ik} - \varepsilon^p_{ik} + z_{ik})e_{kj}),$$

$$\frac{Dp_{ij}}{dt} = \varepsilon^p_{ij} - p_{is}\varepsilon^p_{sj} - \varepsilon^p_{is}p_{sj}, \qquad \frac{Dn_{ij}}{Dt} = \frac{dn_{ij}}{dt} - r_{ik}n_{kj} + n_{ik}r_{kj},$$

$$\varepsilon_{ij} = \frac{1}{2}\,(v_{i,j} + v_{j,i}), \qquad v_i = \frac{\partial u_i}{\partial t} + u_{i,j}v_j, \qquad u_{i,j} = \frac{\partial u_i}{\partial x_j},$$

$$z_{ij} = A^{-1}[B^2(\varepsilon_{ik}e_{kj} - e_{ik}\varepsilon_{kj}) + B(\varepsilon_{ik}e_{kt}e_{tj} - e_{ik}e_{kt}\varepsilon_{tj}) + e_{ik}\varepsilon_{kt}e_{ts}e_{sj} - e_{ik}e_{kt}\varepsilon_{ts}e_{sj}], \tag{1.1}$$

$$A = 8 - 8E_1 + 3E_1^2 - E_2 - E_1^3/3 + E_3/3, \qquad B = 2 - E_1,$$

$$E_1 = e_{jj}, \qquad E_2 = e_{ij}e_{ji}, \qquad E_3 = e_{ij}e_{jk}e_{ki},$$

$$r_{ij} = \omega_{ij} + z_{ij}, \qquad \omega_{ij} = (v_{i,j} - v_{j,i})/2.$$

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Here u_i and v_i are the components of the displacement vectors and velocities of points of the medium; D/Dt is the objective derivative of the tensors with respect to time, and r_{ij} is the rotation tensor, whose components z_{ij} depend on the reversible strains and strain rates. In the equation of variation of the tensor p_{ij}, the source ε_{ij}^p should be referred to, as in the classical theory, as the plastic strain-rate tensor components. According to Eqs. (1.1), during relaxation ($\varepsilon_{ij}^p = 0$), the irreversible strain tensor components vary in the same manner as in rigid body motion. The components of the total Almansi strain tensor d_{ij} are expressed in terms of its components e_{ij} and p_{ij} as

$$d_{ij} = e_{ij} + p_{ij} - e_{ik}e_{kj}/2 - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{ks}e_{sj}. \tag{1.2}$$

The stresses in the medium are completely determined by reversible strains and are related to them, according to the laws of thermodynamics for incompressible media, by the equations

$$\sigma_{ij} = \begin{cases} -p\delta_{ij} + \dfrac{\partial W}{\partial d_{ik}}(\delta_{kj} - 2d_{kj}), & p_{ij} \equiv 0, \\ -p_1\delta_{ij} + \dfrac{\partial W}{\partial e_{ik}}(\delta_{kj} - e_{kj}), & p_{ij} \neq 0. \end{cases} \tag{1.3}$$

Here p and p_1 are additional hydrostatic pressures. Assuming that the medium is isotropic, we write the elastic potential W as

$$W = -2\mu J_1 - \mu J_2 + bJ_1^2 + (b-\mu)J_1J_2 - \chi J_1^3 + \dots\ ,$$

$$J_k = \begin{cases} L_k, & p_{ij} \equiv 0, \\ I_k, & p_{ij} \neq 0, \end{cases} \tag{1.4}$$

$$L_1 = d_{kk}, \quad L_2 = d_{ik}d_{ki}, \quad I_1 = e_{kk} - e_{sk}e_{ks}/2, \quad I_2 = e_{st}e_{ts} - e_{sk}e_{kt}e_{ts} + e_{sk}e_{kt}e_{tn}e_{ns}/4$$

(μ, b, and χ elastic constants of the medium). This choice of the invariants I_1 and I_2 of the reversible strain tensor ensures passage to the limit from the second dependence in (1.3) to the first dependence as the irreversible strains tend to zero.

We assume that irreversible strains are accumulated in the material when the stress state reaches the loading surface, which is the plastic potential, by virtue of the Mises maximum principle. As such a surface, we use the Tresca plasticity condition extended to the case of deformation with allowance for viscosity [6, 7]:

$$\max|\sigma_i - \sigma_j| = 2k + 2\eta \max|\varepsilon_k^p|. \tag{1.5}$$

Here k is the yield limit, η is the viscosity, and σ_i and ε_k^p are the principal values of the plastic stress and strain rate tensors, respectively.

The irreversible strain rates are related to the stresses by the associated plastic flow law

$$\varepsilon_{ij}^p = \lambda\frac{\partial f}{\partial \sigma_{ij}}, \qquad f(\sigma_{ij}, \varepsilon_{ij}^p) = k, \qquad \lambda > 0. \tag{1.6}$$

2. Elastic Equilibrium. Let an elastoviscoplastic material, whose properties are described by the relations given above, fill the space between two cylindrical matrices with rigid walls. We consider the deformation of this material in the case of rotation of the inner rigid cylinder of radius $r = r_0$ and the motionless outer cylinder of radius $r = R_0$. Thus, in cylindrical coordinates (r, φ, z), the boundary condition is written as

$$\boldsymbol{u}\Big|_{r=R_0} = 0. \tag{2.1}$$

We assume that, in the case considered, all points of the medium, including the boundary points, move on circles. Then, the components of the displacement vector have the form

$$u_r = r(1-\cos\theta), \qquad u_\varphi = r\sin\theta,$$

where $\theta = \theta(r, t)$ is the central angle of twisting.

An increase in the angle θ first leads only to elastic deformation. Once the angle reaches a certain value $\theta_0 = \theta(t_0)$, plastic flow begins in the vicinity of the inner rigid wall. Next, setting $t_0 = 0$, we calculate the parameters of the stress–strain state at this time.

In the case considered, the nonzero components of the Almansi tensor are the following:

$$d_{rr} = -\frac{1}{2}\,g^2, \qquad d_{r\varphi} = \frac{1}{2}\,g, \qquad g = r\,\frac{\partial\theta}{\partial r}. \tag{2.2}$$

According to relations (1.3) and (1.4), the stress components, to within strain terms of the second order, are calculated by the formulas

$$\sigma_{rr} = \sigma_{zz} = -(p+2\mu) - (b+\mu)g^2/2 = -s,$$
$$\sigma_{\varphi\varphi} = -s + \mu g^2, \qquad \sigma_{r\varphi} = \mu g. \tag{2.3}$$

Using the boundary condition (2.1) and the plasticity condition in the form

$$(\sigma_{rr} - \sigma_{\varphi\varphi})^2 + 4\sigma_{r\varphi}^2 = 4k^2,$$

from the equilibrium conditions

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \qquad \frac{\partial\sigma_{r\varphi}}{\partial r} + 2\,\frac{\sigma_{r\varphi}}{r} = 0$$

we find the angle of rotation θ_0 at which plastic flow begins:

$$\theta_0 = \frac{k}{2\mu}\Big(1 - \frac{r_0^2}{R_0^2}\Big). \tag{2.4}$$

Under elastic equilibrium conditions, the stress components are defined by the relations

$$\sigma_{rr} = \sigma_{zz} = \frac{k^2}{4\mu}\Big(1 - \frac{r_0^4}{r^4}\Big) + \sigma_0, \quad \sigma_{\varphi\varphi} = \frac{k^2}{4\mu}\Big(1 + 3\frac{r_0^4}{r^4}\Big) + \sigma_0, \quad \sigma_{r\varphi} = -k\,\frac{r_0^2}{r^2},$$

where σ_0 is the value of the stress component σ_{rr} on the surface $r = r_0$ at the moment of the beginning of plastic flow. We can set $\sigma_0 = 0$.

From relation (1.2), we find the following relations necessary for the further calculations:

$$e_{r\varphi} = d_{r\varphi} = -\frac{1}{2}\,\frac{k}{\mu}\,\frac{r_0^2}{r^2}, \qquad e_{rr} = -\frac{3}{2}\,e_{r\varphi}^2, \qquad e_{\varphi\varphi} = \frac{1}{2}\,e_{r\varphi}^2. \tag{2.5}$$

3. Irreversible Deformation. Beginning at the time $t = t_0 = 0$, an increase in the angle of rotation in the vicinity of the inner rigid cylinder leads to the development of a region of viscoplastic flow $r_0 \le r \le r_1(t)$ [$r_1(t)$ is the moving boundary of the region of plastic flow, which separates it from the zone of elastic deformation $r_1(t) \le r \le R_0(t)$].

According to relations (1.1) and (1.2), the kinematics of the medium is defined by the relations

$$u_r = r(1 - \cos\theta(r,t)), \qquad u_\varphi = r\sin\theta(r,t),$$
$$v_\varphi = r\,\frac{\partial\theta}{\partial t}, \qquad \varepsilon_{r\varphi} = \frac{1}{2}\Big(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\Big) = \frac{\partial d_{r\varphi}}{\partial t} = \frac{1}{2}\,r\,\frac{\partial^2\theta}{\partial r\,\partial t},$$
$$\varepsilon_{r\varphi} = \varepsilon_{r\varphi}^e + \varepsilon_{r\varphi}^p = \frac{\partial e_{r\varphi}}{\partial t} + \frac{\partial p_{r\varphi}}{\partial t}, \tag{3.1}$$
$$\varepsilon_{rr}^p = \frac{\partial p_{rr}}{\partial t} + 2p_{r\varphi}(r_{\varphi r} + \varepsilon_{r\varphi}^p), \qquad \varepsilon_{\varphi\varphi}^p = \frac{\partial p_{\varphi\varphi}}{\partial t} + 2p_{r\varphi}(r_{r\varphi} + \varepsilon_{r\varphi}^p),$$
$$\varepsilon_{rr}^p = -\varepsilon_{\varphi\varphi}^p = -2\varepsilon_{r\varphi}^p e_{r\varphi}.$$

Integration of the equilibrium equations (quasistatic approximation) in the region of reversible deformation using condition (2.1) yields

$$\sigma_{r\varphi} = \frac{c(t)}{r^2}, \qquad \theta(r,t) = \frac{c(t)}{2\mu}\Big(\frac{1}{R_0} - \frac{1}{r^2}\Big), \tag{3.2}$$

where $c(t)$ is an unknown function of integration.

From the second relation in (1.3), the stress components in the region of viscoplastic flow $r_0 \le r \le r_1(t)$ are expressed as

$$\sigma_{rr} = \sigma_{zz} = -(p_1 + 2\mu) - 2(b+\mu)e_{r\varphi}^2 = -s_1(t),$$
$$\sigma_{\varphi\varphi} = -s_1(t) + 4\mu e_{r\varphi}^2, \qquad \sigma_{r\varphi} = 2\mu e_{r\varphi}. \tag{3.3}$$

In the derivation of expressions (3.3), we used the kinematic relations (2.5). At the same time, integration of the equilibrium equations yields

$$\sigma_{r\varphi} = \frac{m(t)}{r^2}, \qquad e_{r\varphi} = \frac{m(t)}{2\mu r^2}. \tag{3.4}$$

From the conditions of continuity of the stress components on the boundary of the region of elastoplastic flow $r = r_1(t)$, it follows that

$$m(t) = c(t), \qquad s(t) = s_1(t).$$

The plastic flow condition (1.5) is written as

$$\sigma_{r\varphi}^2 - (k + \eta|\varepsilon_{r\varphi}^p|)^2 = 0. \tag{3.5}$$

The associated plastic flow law (1.6) and condition (3.5) leads to

$$\sigma_{r\varphi} = -k + \eta\varepsilon_{r\varphi}^p, \qquad \lambda = -\varepsilon_{r\varphi}^p/(k - \eta\varepsilon_{r\varphi}^p). \tag{3.6}$$

Using (3.4) and (3.6), we can calculate the plastic strain rate

$$\varepsilon_{r\varphi}^p = \frac{1}{\eta}\Big(\frac{c(t)}{r^2} + k\Big).$$

Taking into account the second equality in (3.4) and using the kinematic relations (3.1) and the condition of continuity of the function $\theta(r,t)$ on the boundary of the region of viscoplastic flow $r = r_1(t)$, for the region of irreversible deformation, we obtain

$$\theta(r,t) = \frac{c(t)}{2\mu}\Big(\frac{1}{R_0^2} - \frac{1}{r^2}\Big) + \frac{c_1(t)}{\eta}\Big(\frac{1}{r_1^2(t)} - \frac{1}{r^2}\Big) + \frac{2kt}{\eta}\ln\frac{r}{r_1(t)}, \qquad c_1(t) = \int c(t)\,dt.$$

From the condition of continuity of the derivative $\partial\theta/\partial r$ on the boundary $r = r_1(t)$ and the loading condition on the boundary $r = r_0$, we calculate the functions $c(t)$ and $c_1(t)$ and obtain the ordinary differential equation for $r_1(t)$:

$$c_1(t) = -ktr_1^2, \qquad c(t) = -k(r_1^2 + 2r_1\dot r_1 t), \qquad \dot r_1 = \frac{dr_1}{dt},$$
$$\dot r_1 = \Big[\frac{kr_1^2}{2\mu}\Big(\frac{1}{R_0^2} - \frac{1}{r_0^2}\Big) + \frac{kt}{\eta}\Big(1 - \frac{r_1^2}{r_0^2} - 2\ln\frac{r_0}{r_1}\Big) + \theta(r_0,t)\Big] \Big/ \Big[\frac{ktr_1}{\mu}\Big(\frac{1}{r_0^2} - \frac{1}{R_0^2}\Big)\Big].$$

The development of the zone of viscoplastic flow $r_1(\tau) = r_1(t)/R_0$ in time ($\tau = \alpha t$) for the values of the constants $\alpha\eta/\mu = 0.001$, $r_0/R_0 = 0.5$, and $k/\mu = 0.006\,21$ is shown in Fig. 1. As the angle of rotation increases [for the numerical solution, we used the linear law $\theta(r_0,t) = \theta_0(1+\alpha t)$], the function $r_1(t)$ asymptotically approaches a certain value dependent on the properties of the material.

The obtained function $r_1(t)$ is used to determine the function $\theta(r,t)$, the stresses, and the total and reversible strains in both the region of reversible deformation and the region of viscoplastic flow. According to formula (1.2), in which the total strains are divided into reversible and irreversible, the plastic strain components are defined by the relations

$$p_{r\varphi} = \frac{kt}{\eta}\Big(1 - \frac{r_1^2}{r^2}\Big), \qquad p_{\varphi\varphi} = 2e_{r\varphi}p_{r\varphi}, \qquad p_{rr} = 2d_{r\varphi}(e_{r\varphi} - d_{r\varphi}).$$

4. Relaxation and Flow for Rotation of the Cylinder in the Opposite Direction. For the stop of the inner cylinder ($\theta = \theta_1$) at a certain time $t = t_1$, the boundary of the region of viscoplastic flow is determined by the value $r_1 = r_1(t_1)$. If the angle of rotation is not increased further, this value does not change. In this case, the strain components, and, hence, the stress components, also remain unchanged. If the deformation process is completed, such strains and stresses are residual.

Let us elucidate how the stress–strain state changes if the inner cylinder is rotated in the opposite direction beginning at the time $t = t_1$ (or any time $t > t_1$).

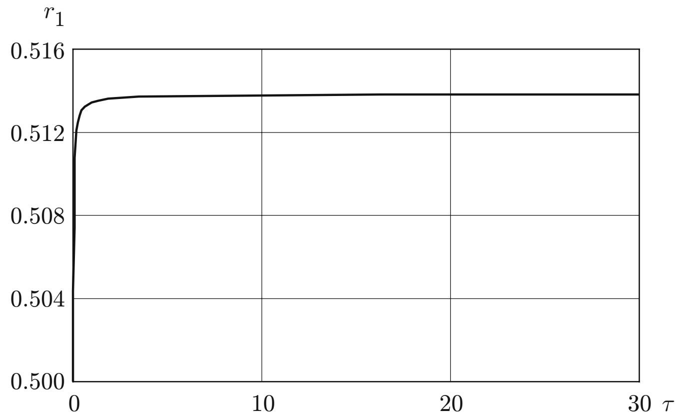

Fig. 1. Boundary $r_1(\tau)$ of the region of viscoplastic flow for rotation of the inner cylinder ($r_0/R_0 = 0.5$).

For $\theta < \theta_2$, only reversible deformation occurs in the material. Beginning at the time $t = t_2$, the stress state in the vicinity of the inner rigid cylinder reaches the loading surface:

$$\sigma_{r\varphi}(r_0) = k. \tag{4.1}$$

In other words, the stress component $\sigma_{r\varphi} = c(t)/r^2$ first decreases in absolute value, and then [for $\theta(t_*) = \theta_*$], it increases until the plasticity condition (4.1) is satisfied and a new region of plastic flow begins to develop in the vicinity of the inner surface. From the equality $\sigma_{r\varphi}|_{\theta=\theta_*} = 0$, we obtain $c(t_*) = 0$. Hence, if the angle of rotation of the boundary surface $r = r_0$ equals θ_*, the stress $\sigma_{r\varphi}$ is equal to zero not only on this surface but also over the entire region of deformation (in both the region of elastic deformation and the region of viscoplastic flow).

To find the value of θ_* and the value of θ_2, which determines the beginning of plastic flow, it is necessary to solve the problem of elastic equilibrium with accumulated irreversible strains. In the region of reversible deformation $r_1 \le r \le R_0$, the strain and stress components are defined by relations (2.2) and (2.3), and the value of $\theta(r,t)$ is defined by relation (3.2). Taking into account that the plastic strain tensor component $p_{r\varphi}$ does not change before the beginning of plastic flow ($\varepsilon^p_{r\varphi} = 0$), we determine the function $\theta(r,t)$ in the region with accumulated irreversible strains using the condition $d_{r\varphi} = e_{r\varphi} + p_{r\varphi}$ and the condition of continuity of $\theta(r,t)$ for $r = r_1$:

$$\theta(r,t) = \frac{2kt_1}{\eta}\Big(\ln\Big(\frac{r}{r_1}\Big) + \frac{1}{2}\Big(\frac{r_1^2}{r^2} - 1\Big)\Big) + \frac{c(t)}{2\mu}\Big(\frac{1}{R_0^2} - \frac{1}{r^2}\Big). \tag{4.2}$$

We note that, although the components p_{rr} and $p_{\varphi\varphi}$ vary, the irreversible strain tensor remains unchanged. From equalities (4.1), (4.2), and $c(t_*) = 0$, we determine the angles θ_* and θ_2:

$$\theta_* = \frac{2kt_1}{\eta}\Big(\ln\Big(\frac{r_0}{r_1}\Big) + \frac{1}{2}\Big(\frac{r_1^2}{r_0^2} - 1\Big)\Big), \qquad \theta_2 = \theta_* - \frac{k}{2\mu}\Big(1 - \frac{r_0^2}{R_0^2}\Big). \tag{4.3}$$

To determine the stress component of the equilibrium equation with a further decrease in the angle θ, it is necessary to integrate in the three regions: the region of reversible deformation $r_1 \le r \le R_0$, the region with the unchanged irreversible strain tensor $r_2(t) \le r \le r_1$, and the regions of plastic flow $r_0 \le r \le r_2(t)$. In the first two regions, as above, the stress components and the function $\theta(r,t)$ are defined by relations (3.2), (3.4) and (4.2), in which the function $c(t)$ needs to be replaced by its current value $x(t)$. For the region of plastic flow $r_0 \le r \le r_2(t)$, using relations (3.4) and the plasticity condition (3.5), we obtain

$$\sigma_{r\varphi} = k + \eta\varepsilon^p_{r\varphi}, \qquad \varepsilon^p_{r\varphi} = \frac{1}{\eta}\Big(\frac{x(t)}{r^2} - k\Big). \tag{4.4}$$

Using the kinematic relations (3.1), the condition of continuity of $\theta(r,t)$ for $r = r_2(t)$, and (4.4), for the region of plastic flow we obtain

$$\theta(r,t) = \frac{x(t)}{2\mu}\Big(\frac{1}{R_0^2} - \frac{1}{r_1^2}\Big) + \frac{kt_1}{\eta}\Big(\ln\Big(\frac{r_2(t)}{r_1}\Big) + \frac{r_1^2}{r_2^2(t)} - 1\Big) - \frac{1}{\eta}\Big(2kt\ln\Big(\frac{r}{r_2(t)}\Big) + x_1(t)\Big(\frac{1}{r^2} - \frac{1}{r_2^2(t)}\Big)\Big),$$

$$x_1(t) = \int x(t)\,dt.$$

Using the condition of continuity of the function $\partial\theta/\partial r$ for $r = r_2$, we determine the unknown functions $x(t)$ and $x_1(t)$ and obtain the following differential equation for $r_2(t)$:

$$x_1(t) = k(t_1 + t)r_2^2 - kt_1 r_1^2, \qquad x(t) = 2kr_2\dot{r}_2(t_1 + t) + kr_2^2,$$

$$\theta(r_0, t) = \frac{2kr_2\dot{r}_2(t_1 + t) + kr_2^2}{2\mu}\Big(\frac{1}{R_0^2} - \frac{1}{r_0^2}\Big) + \frac{kt_1}{\eta}\Big(2\ln\Big(\frac{r_2}{r_1}\Big) + \frac{r_1^2}{r_2^2} - 1\Big)$$
$$- \frac{1}{\eta}\Big[2kt\ln\Big(\frac{r_0}{r_2}\Big) + k((t_1 + t)r_2^2 - t_1 r_1^2)\Big(\frac{1}{r_0^2} - \frac{1}{r_2^2}\Big)\Big]. \qquad (4.5)$$

Equation (4.5) needs to be solved before the time $t = t_3$, at which the surface $r_2(t_3)$ reaches the surface $r = r_1$.

Beginning at the time $t = t_3$, two regions remain in the material: the region of viscoplastic flow $r_0 \le r \le r_2 = r_1$ and the region of reversible deformation $r_1 = r_2 \le r \le R_0$. To trace the further motion of the boundary of the region of viscoplastic flow $r_2(t)$, it is necessary, as above, to determine the stress–strain parameters in both regions by integrating the equilibrium equations. In the region of reversible deformation, the function $\theta(r,t)$ is obtained from relation (3.2) [with the current value $x(t)$ of the function $c(t)$]. In the region of viscoplastic flow, using kinematic relations (3.1), relations (4.4) and the condition of continuity of $\theta(r,t)$ for $r = r_2(t)$, we obtain

$$\theta(r,t) = \frac{x(t)}{2\mu}\Big(\frac{1}{R_0^2} - \frac{1}{r^2}\Big) + \frac{2kt}{\eta}\ln\Big(\frac{r_2}{r}\Big) + \frac{x_1(t)}{\eta}\Big(\frac{1}{r_2^2} - \frac{1}{r^2}\Big).$$

To find the functions $x(t)$ and $x_1(t)$, it is necessary to use the condition of continuity of the function $\partial\theta/\partial r$ for $r = r_2(t)$. As a result, we obtain

$$x_1(t) = ktr_2^2, \qquad x(t) = \dot{x}_1(t) = kr_2^2 + 2ktr_2\dot{r}_2. \qquad (4.6)$$

The displacement vector components [the function $\theta(r,t)$] should be continuous at any time. Hence, at the time $t = t_3$, the continuity condition should be satisfied for the functions $x_1(t)$ and $x(t)$. A comparison of relations (4.5) and (4.6) shows that the function $x(t)$ can be continuous only if $\dot{r}_2(t) = 0$. Thus, from the time $t = t_3$ the surface $r_2(t)$ reaches the original surface which bounds the region with accumulated irreversible strain, the region of viscoplastic flow does not develop further in spite of an increase in the angle of rotation. In this case, plastic strain components change in the region $r_0 \le r \le r_2$, remaining equal to zero on the boundary of the elastoviscoplastic flow $r = r_2$. As in the case described in Sec. 3, for the stop of the inner cylinder at any time $t > t_3$, the strain and stress components remain unchanged.

5. Deformation for Rotation of the Outer Cylindrical Surface. We consider the deformation of an elastoviscoplastic material upon rotation of the outer rigid cylinder with the inner cylinder remaining motionless:

$$\boldsymbol{u}\Big|_{r=r_0} = 0. \qquad (5.1)$$

In this case, plastic flow also begins in the vicinity of the inner rigid wall when the stress state reaches the loading surface (1.5). This plasticity condition is written as (4.1). The angle of rotation θ_0 and the stress components at the beginning of plastic flow are found from the condition (5.1) and the plasticity condition (4.1). The value of θ_0 is the same as in the case of rotation of the inner cylinder [see (2.4)], the stress components are calculated by the relations

$$\sigma_{rr} = \sigma_{zz} = \frac{k^2 r_0^4}{4\mu}\Big(\frac{1}{R_0^4} - \frac{1}{r^4}\Big) + \sigma_0, \qquad \sigma_{\varphi\varphi} = \frac{k^2 r_0^4}{4\mu}\Big(\frac{1}{R_0^4} + \frac{3}{r^4}\Big) + \sigma_0, \qquad \sigma_{r\varphi} = k\,\frac{r_0^2}{r^2}.$$

With a further increase in the angle of rotation, the region of viscoplastic flows is defined by the inequalities $r_0 \le r \le r_1(t)$. In the region $r_1(t) \le r \le R_0$, reversible deformation occurs. The kinematics of the medium is defined by relations (3.1). According to the equilibrium equations, in the region of reversible deformation,

$$\theta(r,t) = \frac{x(t)}{2\mu}\Big(\frac{1}{r_0^2} - \frac{1}{r^2}\Big).$$

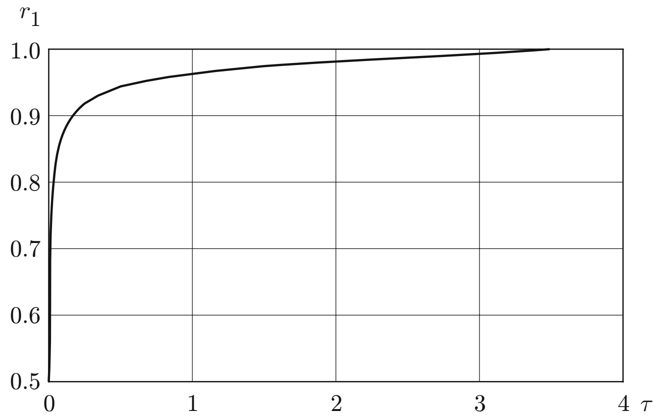

Fig. 2. Boundary $r_1(\tau)$ of the region of viscoplastic flow for rotation of the outer cylinder ($r_0/R_0 = 0.5$).

In the region of viscoplastic flow, this function is determined from relations (4.4), kinematic relations (3.1), and the continuity conditions for $\theta(r,t)$ at $r = r_1(t)$:

$$\theta(r,t) = \frac{2}{\eta}\Big(kt\ln\Big(\frac{r_1}{r}\Big) - \frac{x_1(t)}{2}\Big(\frac{1}{r^2} - \frac{1}{r_1^2}\Big)\Big) + \frac{x(t)}{2\mu}\Big(\frac{1}{r_0^2} - \frac{1}{r^2}\Big).$$

From the condition of continuity of $\partial\theta/\partial r$ for $r = r_1(t)$, we find the functions $x(t)$ and $x_1(t)$ and obtain the following differential equation for the boundary of the region of viscoplastic flow:

$$\theta(R_0,t) = \frac{2r_1\dot{r}_1kt + r_1^2k}{2\mu}\Big(\frac{1}{r_0^2} - \frac{1}{r^2}\Big) + \frac{kt}{\eta}\Big(2\ln\Big(\frac{r_1}{r}\Big) - \frac{r_1^2}{r^2} + 1\Big),$$

$$x_1(t) = ktr_1^2, \qquad x(t) = k(r_1^2 + 2r_1\dot{r}_1t).$$

Figure 2 shows the development of the region of viscoplastic flows for the same linear law of loading and the same constants as in Fig. 1. Unlike in the case of rotation of the inner cylinder, the boundary of the region of viscoplastic flow does not have an asymptote and, with time, it reaches the outer boundary surface $r = R_0$.

In the case of stop of the outer cylinder and its rotation in the opposite direction, the same effects are observed as for the rotation of the inner cylinder. The value of θ_2 for which the stress state for the rotation of the outer cylinder in the opposite direction reaches the loading surface $\sigma_{r\varphi}|_{r=r_0} = -k$ is defined by relation (4.3), in which it is necessary to set

$$\theta_* = \frac{2kt_1}{\eta}\Big(\ln\Big(\frac{r_1}{R_0}\Big) + \frac{1}{2}\Big(1 - \frac{r_1^2}{R_0^2}\Big)\Big).$$

The differential equation for the boundary $r_2(t)$ of the new region of viscoplastic flow becomes

$$\theta = \frac{2}{\eta}\Big(kt\ln\Big(\frac{r}{r_2}\Big) + \frac{kr_2^2(t+t_1) - kt_1r_1^2}{2}\Big(\frac{1}{r^2} - \frac{1}{r_2^2}\Big)\Big) + \frac{kt_1}{\eta}\Big(2\ln\Big(\frac{r_1}{r_2}\Big) - \frac{r_1^2}{r_2^2} + 1\Big)$$

$$+ \frac{2kr_2\dot{r}_2(t+t_1) + kr_2^2}{2\mu}\Big(\frac{1}{r_0^2} - \frac{1}{r^2}\Big).$$

From the moment the boundary $r_2(t)$ reaches the surface r_1, the region of viscoplastic flows does not develop.

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